

The problem of selecting exact and approximate models of heat transfer is analyzed for the solution of inverse problems.

Inverse problem methods held a strong place in heat transfer research practice. The development of regularizing algorithms afforded the possibility of overcoming the incorrectness of inverse problem formulation and of their successful application to process measurements and raise the informativity of experiments on this basis [1, 2]. Extremely valuable to the formulation of inverse, just as for direct, problems is the selection of the mathematical model relating the cause and effect characteristics. As a rule, a certain number of models, differing by the completeness and accuracy of the process description on the one hand, and by the complexity level on the other [3, 4], can be compiled for the very same process. Selection of the model structure (structural identification) is the first step in mathematical formulation of the problem and can be formalized in the form of successive complication of the model and its verification for adequacy to the real process [4]. Such a sorting is naturally determined mostly by the subjective experience of the researcher. The quality of the selected mathematical model is characterized by compromising different, often contradictory, requirements. Then the requirement of adequacy of the model to the real process is just one of the conditions of its efficient utilization, but is by far not the only one. Such properties of the mathematical model as its simplicity, machine time expenditure in a numerical realization on an electronic computer, especially during execution of a large series of computations or during solution of problems in a real time scale are quite important, governing in a number of cases.

Since model insufficiencies are extensions of their virtues, a situation is completely possible when one of the models of a set satisfies the researcher for some reason or other. In other words, the value of the functional

$$J = J_1 + J_2 + J_3 + \dots, \quad (1)$$

where J_1 is the criterion of model adequacy to the real process, J_2 is the expenditure of manual labor in compiling the algorithm for the computation and the program and J_3 is the machine time expenditure, etc., turns out to be insufficiently small, no matter what model would have been selected as the working model.

Thus, according to the example presented in [5], the solution of the problem of restoring the boundary heat flux density by a gradient method with the distorting action of the thermocouple taken into account requires several hours of machine time even for a sufficiently rough discretization of the space-time domain, which causes great difficulty in the necessity to process a number of measurements. In this case, the adequacy demand has been satisfied but with substantial machine time expenditure. On the other hand, the solution of this problem under conditions of thermocouple spotting results in lowering of the machine time expenditure by two orders but the error in restoration of the desired function reaches several tens of percents.

Nevertheless when going from model to model the diminution of one of the components of the sum (1) will result in an automatic increase of the other (or others). And though every researcher ranks the components of (1) in his own manner on the basis of personal experience and understandably taking account of external conditions, the example presented shows that situations exist when one of the models are themselves in a state to satisfy simultaneously the imposed demands. In such cases, utilization of a two-model algorithm proposed in [6] and permitting reduction of the tedium of the calculations and the machine time expenditure (i.e.,

diminution of the values of J_2 and J_3) can turn out to be useful in the solution of the inverse heat conduction problem.

The crux of this approach is that the initial formulation of the inverse heat conduction problem

$$Au = f_\delta \quad (2)$$

should be reformulated as follows

$$\tilde{A}^{(k+1)} u = f_\delta \frac{\tilde{A}^{(k)} u}{A u}, \quad k = 0, 1, 2, \dots, \quad (3)$$

where A , \tilde{A} are, respectively, the adequate (exact) and approximate models of the heat transfer process connecting the identical restorable causal characteristic u and the input information f_δ , as a rule, to the temperature at an inner point, known in the general case with an error that fluctuates in the majority of cases.

It should be noted that a method is proposed in [7] for restoring the boundary condition with a correction of the approximate model by solving the direct problem for the adequate model based on utilization of the Green's function, however, its domain of application is constrained to linear formulations.

At the same time the procedure (3) actually performs an iteration by correcting the input information by adjusting it to the approximate model, utilization of the adequate model hence occurs just during the stage of solving the direct problem while the inverse problem is solved intrinsically by the approximate method in each iteration (3). In the general case the models A and \tilde{A} can here be nonlinear.

The example of restoring the boundary heat flux density with the perturbing action of the heat sensor taken into account, presented in [6], indicates that four-five iterations according to the models in conformity with the algorithm (3) will be sufficient to obtain a solution with good accuracy. However, computations show that the number of necessary iterations can be diminished if the initial approximation $u^{(0)}$ is not selected arbitrarily but from the condition $\tilde{A}^{(0)} u^{(0)} = f_\delta$, i.e., it is determined from the solution of the inverse problem for the approximate model without correction of the input data. Such a selection of $u^{(0)}$ affords the possibility of solving the inverse heat conduction problem in the formulation (3) in just one, at most two, iterations and thereby reduce the machine time expenditure still more.

The machine time savings because of using the two-model algorithm (3) when solving the inverse heat conduction problem by a gradient method can be estimated approximately by the quantity $3N_g/N_m$, where N_g is the number of steps in the gradient descent to the point of the minimum residual, N_m is the number of iterations according to the models, and the factor 3 appears because a computation by an adequate model in the one-model formulation (2) is performed at least three times at each step of the descent (the direct, the adjoint problem and the problem in increments). It is assumed in such an estimate of the machine time savings that the solution of the direct problem for the adequate model takes up the main time in each iteration (3). Therefore, even when using the rapidly convergent method of conjugate gradients, when the main structural singularities of the desired function appear after four-six steps in the majority of cases, the application of the two-model algorithm will result in a fifteenfold reduction in the machine time expenditure. This index will be still higher for restoration of causal characteristics of complex behavior, however in this case the necessary condition is smallness of the variance of the random error in the input data [6].

It can also be noted that another method exists for correcting the input information that consists of appending a certain correction, the two-model algorithm is here realized in the form

$$\tilde{A}^{(k+1)} u = f_\delta + \Delta f_k, \quad k = 0, 1, 2, \dots, \quad (4)$$

$$\Delta f_k = \tilde{A}^{(k)} u - A u. \quad (5)$$

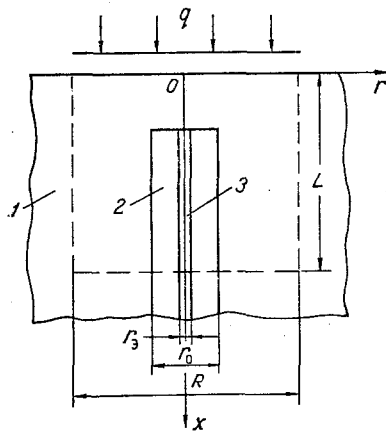


Fig. 1

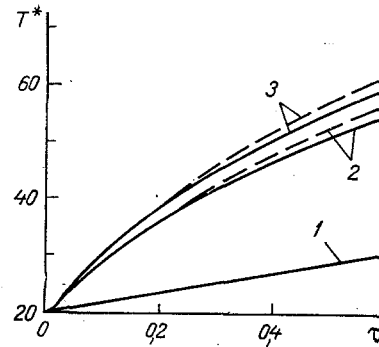


Fig. 2

Fig. 1. Arrangement of the heat sensor in the body; 1) body; 2) thermocouple electrodes; 3) insulation.

Fig. 2. Computed temperature at the measurement point for $q = 8 \cdot 10^5 \text{ W/cm}^2$ (solid lines are variable thermophysical characteristics, dashes are constant thermophysical characteristics): 1) model a); 2) model b); 3) model c). T^* , °C; τ , c.

It is clear than in the general case the efficiency of the procedure (4) and (5) in the sense of its convergence rate will be distinct from the analogous characteristic of the procedure (3). Thus it is easy to show that if the relationship $\tilde{A}u = (1 + \epsilon)Au$, holds, where ϵ is a constant independent of u , then the algorithm (3) yields the exact solution of the inverse heat conduction problem in the first iteration while the additive correction (4) and (5) assures convergence to the exact solution according to the law $|u/u^{(k)*} - 1| = \epsilon^{k+1}/(1 + \epsilon)^{k+1}$, i.e., it converges more slowly than (3). Since the quantity ϵ depends weakly on the form of the function u quite often and can therefore be considered a fully stable parameter characterizing the error of model \tilde{A} with respect to model A [6], then utilization of the procedure (3) to solve the inverse heat conduction problem is more preferable in these cases.

Both the exact and the approximate models were considered given a priori in [6] for the realization of the two-model algorithm in a specific example of restoration of the boundary heat flux density while the main problem is namely the successful selection of these models. It is clear that the approximate model should be sufficiently simple on one hand, i.e., contain less information about the physics of the process and assure less difficulties in the numerical realization than the exact model. On the other hand, the discontinuity between the approximate and exact models should not be too great so that a high rate of convergence of two-model algorithm would be assured. Hence, quite contradictory demands are imposed on the quality of the approximate model. Selection of adequate model should be governed by its conformity to the real process and, in the long run, by the level of the error in restoring the causal characteristic as well as the labor-intensity of its algorithmization in the direct problem.

Let us consider this problem in an example of a problem to restore the time-dependent boundary heat flux density according to readings of a chromel-alumel thermocouple located within a steel body (Fig. 1). Such problems occur in the research on casting, cutting, contact heat transfer processes, etc. In conformity with the completeness of taking account of the physical heat transfer phenomena we present a list of certain possible models connecting the boundary heat flux density and the body temperature:

a) Concentrated nonstationary thermal model ($T = T(\tau)$)

$$\rho c \frac{dT}{d\tau} = \frac{1}{L} q(\tau), T(0) = T_0;$$

b) Distributed nonstationary thermal model ($T = T(x, \tau)$)

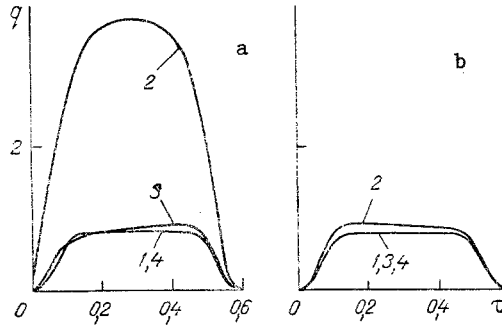


Fig. 3. Restoration of the boundary heat flux density for the concentrated (a) and distributed (b) models selected as approximate (input data perturbed according to a normal law $\sigma_T = 3\% T_{\max}^*$): 1) Actual solution; 2) initial approximation ($u = A^{-1}f_G$); 3, 4) respectively, the first and second iterations of the procedure (3) $q, 10^6$ W/cm²; τ, c .

$$\rho c \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right), \quad T(x, 0) = T_0,$$

$$-\lambda \frac{\partial T(0, \tau)}{\partial x} = q(\tau), \quad \frac{\partial T(L, \tau)}{\partial x} = 0;$$

c) Distributed nonstationary thermal model taking the thermal sensor into account ($T = T(x, r, \tau)$)

$$\rho c \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \lambda \frac{\partial T}{\partial r} \right), \quad T(x, r, 0) = 0,$$

$$-\lambda \frac{\partial T(0, r, \tau)}{\partial x} = q(\tau),$$

$$\frac{\partial T(L, r, \tau)}{\partial x} = \frac{\partial T(x, 0, \tau)}{\partial r} = \frac{\partial T(x, R, \tau)}{\partial r} = 0;$$

d) Distributed nonstationary thermal model with the thermal sensor and the finite rate of heat propagation taken into account ($T = T(x, r, \tau)$)

$$\rho c \left(\frac{\partial T}{\partial \tau} + \tau_0 \frac{\partial^2 T}{\partial \tau^2} \right) = \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \lambda \frac{\partial T}{\partial r} \right),$$

$$T(x, r, 0) = T_0, \quad \frac{\partial T(x, r, 0)}{\partial \tau} = 0, \quad -\lambda \frac{\partial T(0, r, \tau)}{\partial x} = q(\tau),$$

$$\frac{\partial T(L, r, \tau)}{\partial x} = \frac{\partial T(x, 0, \tau)}{\partial r} = \frac{\partial T(x, R, \tau)}{\partial r} = 0.$$

This series can be continued even further, say, because of taking account of the nonlinearity in the thermophysical characteristics (TPC), anisotropies, etc.

It is clear that the quantitative contribution of each of the factors mentioned to the total thermal pattern will be distinct. Thus, taking account of the distributivity of the heat transfer results in a change of several times in the temperature field, taking account of the real thermal sensor geometry 10-30%, taking account of TPC nonlinearity 5-10% (Fig. 2). The influence of the remaining factors on the temperature in the body equals zero in practice.

Therefore, the model c), i.e., the distributed nonlinear nonstationary model of heat conductivity with the thermal sensor taken into account, can be selected as the adequate ther-

mal model for solving the inverse heat conduction problem in this case, as taking all essential factors most completely into account.

The situation becomes the following with the election of the approximate model. Results of solving the inverse heat conduction problem by a two-model method (3) that illustrates the efficiency of using certain of the presented models as approximate, are represented in Fig. 3. Application of the linear distributed model b) is slightly more complicated in realization than the concentrated model a), assures rapid convergence to the exact solution in just one iteration of the procedure (3). Utilization of the two-dimensional linear model c) as approximate also permits obtaining a solution of the inverse heat conduction problem in one iteration, however, its realization requires a very much greater machine time expenditure than realization of the one-dimensional model b). The linear concentrated model a), being the simplest of all, required two iterations of the procedure (3). Results of applying the nonlinear models a) and b) as approximate are identical to those presented above. The deduction can therefore be made that model b) most efficiently fills the role of an approximate model in the solution of this inverse heat conduction problem. On one hand such a selection assures the maximal rate of convergence, and on the other not too intensive labor for algorithmization and acceptable machine time expenditure.

The realization of a two-model algorithm should thereby be specified as the preliminary stage in performing test computations with the sorting of several approximate model modifications. The model selected as approximate should be the simplest among all the models assuring the most rapid convergence of the procedure (3).

The space-time domain was discretized with the number of steps $n_\tau \times n_x \times n_r = 60 \times 40 \times 30$ during execution of the computations. The numerical solution of two-dimensional problems was performed by a locally one-dimensional method, by factorization in each of the coordinates. The linear concentrated and distributed formulations were converted into Duhamel integrals. The following values of the parameters were selected: $R = 5 \cdot 10^3$ m; $L = 1.2 \cdot 10^{-2}$ m, $r_0 = 0.6 \cdot 10^{-3}$ m, $r_e = 0.15 \cdot 10^{-3}$ m. The thermophysical properties of the materials were taken from [8].

The inverse problem was intrinsically formulated in an external formulation in each iteration of the procedure (3) and was solved by the conjugate gradient method. The cessation of the descent process to the minimum residual point was realized by an additional measurement [9]. The identical useful signal perturbed according to a normal law with a given magnitude of the root-mean-square deviation was taken here as the main and additional information. In application to the results in Fig. 3 utilization of this cessation method yielded the number of steps at which the computation should be terminated $\bar{k} = 6$. In this case application of the two-model algorithm thereby assured an approximately fifteen-fold savings in machine time as compared with the formulation (2).

NOTATION

x, r , cylindrical coordinates, τ , running time; τ_0 , relaxation time; ρ , density; c , specific heat; λ , heat conduction coefficient; L , degree of penetration of thermal perturbations; R , radius of the distorting action of the thermocouple; q , boundary heat flux density; T^* , temperature at the measurement point.

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